THE LINEARITY OF THE MASS-SCALE OF ASTON'S MASS-SPECTROGRAPH

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ABSTRACT. The portion of the mass-distance curve where it is most linear has been investigated, starting from the relationship between mass and distance as obtained by Aston and Fowler from considerations of pure geometry, and as obtained by Pidduck making use of co-ordinate geometry. Treatment of both the expressions leads to the same result, which is identical with that of Pidduck, but is different from that of Aston and Fowler. The mass-distance curve is also plotted and discussed.

STATEMENTS AND RESULTS

The mathematics of Aston's mass-spectrograph is not of much practical importance, since the results obtained by the use of the instrument must be regarded to be only of an empirical character. However, it must have been satisfactory to find that a mathematical analysis of the experimental arrangements did corroborate the empirically obtained results. The chief factor, mathematical or empirical, on which the success of the experiment and the importance and simplicity of its results depend, is that the relationship between the masses, or rather, the ratios of the mass to the charge, of the charged particles concerned, and the distances, from a fiducial point, of the "spectral" lines due to them on the photographic plate, is almost linear over quite a large range. It is the mathematical determination of the region over which the proportionality holds and in which the photographic plate must be placed that forms the subject-matter of this communication.

Aston and Fowler, in their paper[1] dealing with the mathematics of the experiment, show that the mass-scale is approximately linear in the neighbourhood of the region given by $\phi=4\theta$, where θ and ϕ are deflections, in opposite directions, due to the electric and magnetic fields respectively. The relationship between masses and the distances of lines due to them was obtained by Aston and Fowler by geometrical considerations. An analytical solution of the problem is given in Pidduck's Treatise on Electricity,[2] which is much simpler than Aston and Fowler's geometrical treatment. Taking rectangular co-ordinates with the origin at the commencement of the electric field (A, Fig. 1), Pidduck finds that the curve

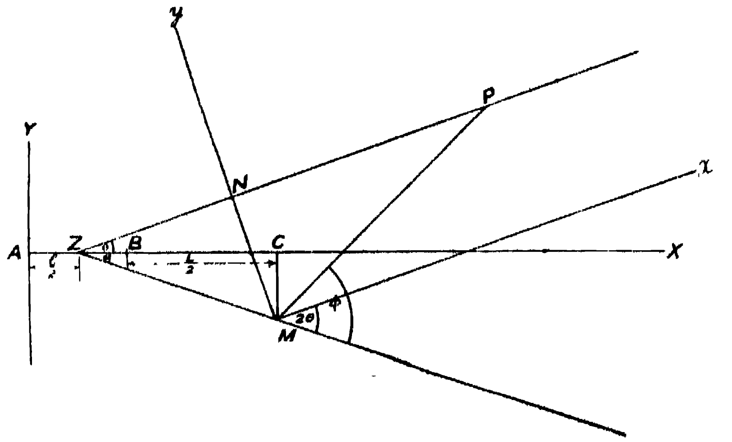

FIGURE 1

The geometry of the mass-spectrograph

AB, the length of the electric field; BC, half that of the magnetic field. Z, the centre of the electric field. AX, AY, the axes as taken in Pidduck's book, Mx, My, the axes as taken by Aston and in this paper.

obtained by plotting m, the mass, or rather, the specific mass, m/e, against x, the distance of the line due to this mass from the origin, has a point of inflexion in the region where the curve and, therefore, the mass-scale would, of course, be nearly linear. The point of inflexion comes out at $x = \frac{5}{4}l + \frac{3}{4}L$, where l and L are respectively the lengths over which the electric and magnetic fields act, the two fields being co-terminus at one of their ends. Then in the relationships

$$\theta = \frac{y}{x - \frac{1}{2}l}, \qquad \phi = \frac{2y}{x - l - \frac{1}{2}L},$$

which hold for focussing the particles having the same m/e, by substituting for x its inflexional value, one finds that the relationship between mass and distance is linear when $\phi = 6\theta$.

Thus the value of ϕ as given in Pidduck's treatise namely "m in this region is nearly proportional to the distance of P from the fiducial point," and that as given in Aston's book,[3] near about which "the mass-scale is approximately linear," are very different. This difference is rather puzzling, for at first sight the two appear to refer to the same region. However, a deeper and closer scrutiny reveals that the two writers refer to different regions. Whereas Pidduck refers to the point of inflexion on the mass-distance curve, Aston, although attempting to find the position where the mass-scale is nearly linear, really determines the point where a straight line from the origin touches this curve. Evidently, the region referred to by Pidduck is the one which is relevant to the problem in hand and not the one found by Aston and Fowler. For, a curve will be approximately linear on either side of a point of inflexion, if there be such a point on the curve ; but round about the point of contact between a curve and the tangent from the origin, the curve need not at all be linear.

The mathematical determination of the position where the relationship between mass and distance may be linear is of no practical importance, as has already been remarked, for the result so obtained is not used in the calculation of the masses. Moreover, on plotting the curve it is found that between the region given by $\phi = 10\theta$ approximately, which is not very far from the origin and that given by $\phi = 2\theta$, which corresponds to the region of the curve at infinity, the curvature is nowhere very pronounced. Thus within comparatively large regions the curve, *i.e.*, the relationship between mass and distance, will be approximately linear, wherever the region may be situated within the limits given above The result, $\phi = 4\theta$, of Aston and Fowler conveys a sense of accuracy which is not true ; whereas the result, $\phi = 6\theta$, of Pidduck gives an impression of uniqueness of solution which is unnecessary.

ASTON AND FOWLER'S MATHEMATICAL EXPRESSION

The expression for r, the distance, measured from a suitable point of the spectral line due to a mass m, as obtained by Aston and Fowler,[3] is

$r = NF/p$ (in Aston and Fowler's notation, p being a constant$=$MN of Figure 1 ; also F of Aston and Fowler is P of this Figure 1)

$$= \frac{m - m_0 + 2 \tan 2\theta \sqrt{mm_0}}{2\sqrt{mm_0} - (m - m_0) \tan 2\theta}, \qquad \ldots \quad (i)$$

where m_0 is another constant and $\sqrt{mm_0} = \cot \frac{\phi}{2}$, θ and ϕ being the electric and magnetic deflections respectively.

Writing $\sqrt{mm_0} = z$, one gets

$$\frac{r}{z^2} = \frac{z^2 - 1 + 2z \tan 2\theta}{z^2\{2z - (z^2 - 1)\tan 2\theta\}}.$$

Aston and Fowler find the region where the above relationship is approximately linear by determining where $\frac{d}{dz}\left(\frac{r}{z^2}\right)$ vanishes. This gives

$$(z \tan 2\theta - 1)\{(3z^2 - 1)\tan 2\theta + z(z^2 - 3)\} = 0.$$

The second factor equated to zero must be considered irrelevant to the problem in hand, for the exact roots of this cubic can be shown to be

$$\phi = \frac{\pi + 4\theta}{3}, \quad \frac{3\pi + 4\theta}{3}, \quad \frac{5\pi + 4\theta}{3},$$

where $$z = \cot \frac{\phi}{2}.$$

With $\theta = 5°$, we are led to

$$\phi = 67°,\ 187°,\ 307° \text{ approximately.}$$

All these values are clearly inapplicable for our purpose. Thus for the problem in hand, Aston and Fowler put

$$z \tan 2\theta - 1 = 0,$$

which gives, $\tan 2\theta = 1/z = \tan \phi/2$, so that for small values of θ and ϕ, $\phi = 4\theta$.

As already remarked, this merely determines where $d/dz(r/z^2) = 0$, which only gives the condition where the tangent from the origin touches the curve.*

To find whether the mass-scale is anywhere linear and, if so, where, we must first investigate whether the above curve has a point of inflexion. To do this, we put in the above relationship between r and m, $\tan 2\theta = a$, $\cot 2\theta = a'$, so that $aa' = 1$; and then determine $d^2 r/dm^2$. Thus

$$r = \frac{m - m_0 + 2a\sqrt{mm_0}}{2\sqrt{mm_0} - (m - m_0)a}.$$

$$\therefore \quad -ar = \frac{m + 2a\sqrt{mm_0} - m_0}{m - 2a'\sqrt{mm_0} - m_0} = 1 + \frac{2(a + a')\sqrt{mm_0}}{m - 2a'\sqrt{mm_0} - m_0}.$$

$$\therefore \quad \frac{-ar - 1}{2(a + a')\sqrt{m_0}} = \frac{\sqrt{m}}{m - 2a'\sqrt{mm_0} - m_0}$$

$$= \frac{A}{\sqrt{m} - k\sqrt{m_0}} + \frac{B}{\sqrt{m} + \frac{1}{k}\sqrt{m_0}} = \frac{A}{P} + \frac{B}{Q},$$

where $k - \frac{1}{k} = 2a'$; $A + B = 1$; $A = Bk^2$; P and Q are obvious.

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* For the tangent at a point, (x, y), is given by

$$(Y - y) - (X - x)\, dy/dx = 0$$

This will go through the origin, $X = Y = 0$, when

$$(y - x)\, dy/dx = 0. \qquad \ldots \quad (1)$$

And $d/dx(y/x) = (1/x) dy/dx - y/x^2$, and vanishes precisely when (1) holds.

Thus dr/dm is proportional to $\left(\frac{A}{P^2}+\frac{B}{Q^2}\right)m^{-\frac{1}{2}}$.

$$\therefore \quad d^2r/dm^2 : al \text{ to } \left(\frac{A}{P^3}+\frac{B}{Q^3}\right)\left(-m^{-1}\right)+\left(\frac{A}{P^2}+\frac{B}{Q^2}\right)\left(-\tfrac{1}{2}m^{-\frac{3}{2}}\right)$$

or
$$m^2\frac{d^2r}{dm^2} : al \text{ to } k^2\frac{3m-k\sqrt{mm_0}}{(\sqrt{m}-k\sqrt{m_0})^3}+\frac{3m+\frac{1}{k}\sqrt{mm_0}}{\left(\sqrt{m}+\frac{1}{k}\sqrt{m_0}\right)^3}.$$

This will be zero when

$$\left(\sqrt{m}-k\sqrt{m_0}\right)^3\left(3m+\frac{1}{k}\sqrt{mm_0}\right)$$
$$+k^3\left(\sqrt{m}+\frac{1}{k}\sqrt{m_0}\right)^3\left(3m-k\sqrt{mm_0}\right)=0,$$

i.e.,
$$3(m/m_0)^2+\frac{1-k^2}{k}(m/m_0)^{\frac{3}{2}}+6m/m_0+3\frac{1-k^2}{k}(m/m_0)^{\frac{1}{2}}-1=0.$$

Now $\quad m/m_0=\cot^2\phi/2,\;$ and $(1-k^2)/k=-2a'=-2\cot 2\theta.$

$\therefore \qquad d^2r/dm^2$ vanishes when

$$3\cot^4\phi/2-2\cot^3\phi/2.\cot 2\theta+6\cot^2\phi/2-6\cot\phi/2.\cot 2\theta-1=0,$$

i.e., when
$$\cot 2\theta=\frac{3\cot^4\phi/2+6\cot^2\phi/2-1}{2\cot^3\phi/2+6\cot\phi/2}$$

$$=\cot\phi+\frac{2}{(2-\cos\phi)\sin\phi}.$$

Thus for small values of θ and ϕ, $d^2r/dm^2=0$, when

$$\frac{1}{2\theta}=\frac{1}{\phi}+\frac{2}{\phi},$$

i.e., when $\phi\simeq 6\theta$.

It can be shown that for this value of ϕ, $d^3r/dm^3\neq 0$. This value of ϕ, therefore, gives the point of inflexion where the relationship between m and r will be linear.

THE ANALYTICAL TREATMENT OF THE PROBLEM

The relation between r and m given above, as found by Aston and Fowler using a pure geometrical method, is complicated and, as will be seen from the foregoing, the finding of d^2r/dm^2 from that expression involves some intricate analysis. By making use of analytical geometry (in the manner of Pidduck), one can obtain more easily a simpler relationship between r and m, and with the help of the new expression get all the information needed.

Thus, following the usual practice, we assume for the purpose in hand that the deflections due to the electric and magnetic fields take place at the respective centres (Z and M, Fig. 1) of the fields, the electric deflections being downwards and the magnetic upwards; and following Aston, take a straight line through M given by $\phi = 2\theta$ as the axis of X. The foci lie on the straight line ZP parallel to MX (Fig. 1). The co-ordinates of a focal point, such as P, will be given by

$$x = \mathrm{NP} = r,$$

$$y = \mathrm{MN} = b.2\theta.$$

where b is the distance between Z and M, the centres of the two fields. It should be remembered that θ and ϕ are assumed to be small.

The co-ordinates of a focus being $(r, 2b\theta)$, we have

$$\theta = y/2b, \text{ and } \phi - 2\theta = y/x = 2b\theta/x.$$

$$\therefore \qquad \phi = 2\theta(b/x + 1).$$

In virtue of the fact that $\theta = k(e/m)(1/v^2)$ and $\phi = k'(e/m)(1/v)$, where e, m and v are respectively the charge, mass and velocity of the particle concerned and k and k' are geometric constants, we have

$$\frac{\phi^2}{\theta} = 4\theta\left(\frac{b}{x} + 1\right)^2 = \frac{k'^2}{k} \cdot \frac{e}{m}$$

$$\therefore \qquad m = \frac{k'^2}{k} \cdot \frac{e}{4\theta} \cdot \left(\frac{x}{b+x}\right)^2$$

or

$$\frac{m}{\mathrm{R}} = \left(\frac{x}{b+x}\right)^2, \qquad \ldots \quad (ii)$$

where R is a constant for a given θ and for the same value of e for all the particles, *i.e.*, for all particles having the same charge.

From the above, it can be easily shown that

$$\frac{d^2 m}{dx^2} = \mathrm{R}\frac{2b(b-2x)}{(b+x)^4},$$

and that the point of inflexion is at $x = b/2$.

Since $\phi = 2\theta\left(\frac{b}{x} + 1\right)$, we have, at the point of inflexion,

$$\phi = 6\theta.$$

The above relationship between m and x is, allowing for a change of axes, the same as that derived by J. J. Thomson in the discussion on isotopes held in the Royal Society on March 3, 1921.[4]

THE CURVE

The complete curve connecting x and m in the equation (*ii*) above is given at A in Fig. 2. In the same figure the curve given by Aston's expression [equation (*i*), Section 2] is also shown at B. The two curves are, as of course they should be, the same, except that the difference in notation gives rise to differences in the positions of the axes and in the values of the constants.

The two ends of the curve asymptotically meet a line parallel to the x-axis, the left end from above and the right end from below. The top of the curve is an exceedingly sharp cusp at infinity. At its lowest portion the curve circles sharply round and then very gradually changing the sign of curvature goes on to meet the asymptote from below.

The curvature at the lowest point is so small that the curve (Fig. 2, A or B) appears to have a cusp at this point as well. In order to exhibit the true nature of the curve in its lowest portion and round about the point of inflexion, the curve must be magnified very considerably, so much so that the resulting figure would be much too large for reproduction. To bring out the nature of the curve at this portion, it has been therefore magnified as well as distorted and shown at C in the right half of the same figure (Fig. 2).

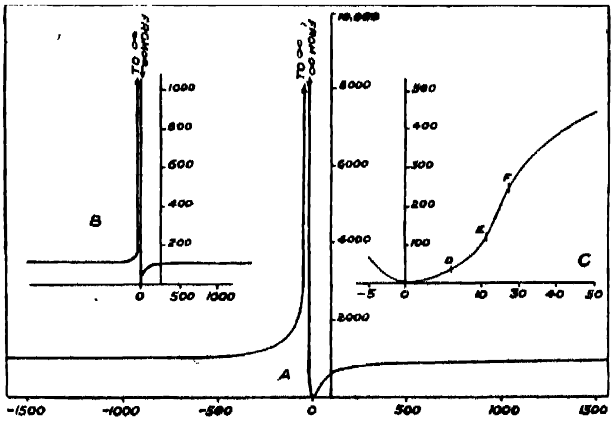

FIGURE 2

The mass-distance curves

In the middle, at A, is shown the plot of the equation,

$$\frac{m}{R}=y=\left(\frac{x}{b+x}\right)^2,$$

taking $b=25$.

On the left, at B, is the plot of the equation,

$$\frac{m}{m_0}=y=1+2\left(\frac{x-0.2}{1+0.2x}\right)^2+2\left(\frac{x-0.2}{1+0.2x}\right)\sqrt{\left(\frac{x-0.2}{1+0.2x}\right)}+1,$$

which is Aston's expression (equation *i*, Section 2) put in a form suitable for plotting (neglecting the negative value of the square root which is irrelevant for present purposes) taking $\tan 2\theta=0.2$. At C is shown the portion of the curve A near the origin, magnified but distorted.

The points marked as D, E, F, on the curve C, Fig. 2, correspond approximately to $\phi=10\theta$, $\phi=6\theta$, and $\phi=4\theta$, respectively. It will be seen that beyond D, the curvature is nowhere very marked, so that for small portions of the curve, the mass-scale would be approximately linear at all points. Beyond F, corresponding to $\phi=4\theta$, however, the change in mass (ordinate) for a given change in the distance from the origin (abscissa) or in other words the distance between two spectral lines due to a certain difference in masses, what may be called the 'dispersion' of the curve, becomes very small.

The deviation from linearity round about various points on the right half of the curve cannot, unfortunately, be shown unless the curve is drawn on an enormously magnified scale This departure from linearity can, however, be easily seen otherwise. Figures, given in Table I, show that the curve is most linear, and at the same time the 'dispersion' is greatest round about the point $\phi=6\theta$, the point of inflexion.

TABLE I

Dispersion and deviation from linearity at various points on the curve $m/R=m'=\{x/(b+x)\}^2$, b being taken $=25$

ϕ/θ	x	M_x	for $dx=\frac{1}{2}$		for $dx=1$		for $dx=2$		for $dx=3$	
			D	d	D	d	D	d	D	d
11	50/9	33.06	9.7	20	9.7	41	9.6	84	9.4	133
10	6.25	40.00	10.2	16	10.2	32	10.1	66	10.0	104
9	50/7	49.38	10.7	12	10.7	24	10.6	48	10.5	75
8	25/3	62.50	11.2	7	11.2	16	11.2	31	11.1	48
7	10	61.61	11.7	3	11.6	7	11.6	15	11.6	16
6	12.5	111.111	11.8	0	11.8	0	11.8	0.2	11.8	1
5	50/3	160.00	11.5	3	11.5	6	11.5	12	11.5	18
4	25	250.00	10.0	5	10.0	10	10.0	18	10.0	33
3	50	444.44	5.9	3	5.9	9	5.9	20	5.9	30
2.5	100	640.00	2.6	0	2.6	8	2.6	14	2.6	21
2.2	250	826.45	0.6	0	0.6	8	0.6	8	0.6	3

Note (1) For $\phi=n\theta$, $x=50/(n-2)$.

,, (2) If $m'_x\times 1000=M_x$, dispersion is taken to be $D=(M_{x+dx}-M_{x-dx})/2dx$.

,, (3) If $M_{x+dx}-M_x=d_1$, $M_x-M_{x-dx}=d_2$, then deviation from linearity is taken to be $(d_1-d_2)/(d_1+d_2)=d'$; $d'\times 1000=d$.

In practice, therefore, the photographic plate may be placed anywhere in the region from $\phi=4\theta$ to $\phi=$ (say) 8θ. The 'fiducial' point will of course be different according to the region chosen, a fiducial point being given by the intersect of the tangent concerned with the axis of x. However, in actual determination of the masses, no use is made of the formula at all, and, therefore, the position of the fiducial point or that of the plate does not really matter. Approximate measurements made on the diagrams given in Aston's book[3] gave the following results:—

From Fig. 9, p. 42, Diagram of the First Mass-Spectrograph, $\theta=3°$, $\phi=21°$ for the middle of the photographic plate.

From Fig. 19, p. 72, Diagram of the Second Mass-Spectrograph, which is specifically indicated to have been given to scale, $\theta=6°$, $\phi=28°$, $33°$, $42°$, for the near end, the middle and the far end, respectively, of the plate.

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